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Inequalities for some classical lattice and continuum systems

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Abstract. Inequalities derived by Holley are applied to the lattice gas model, Ising model and higher order spin systems as well as to continuum analogues of spin-1 lattice systems. As a result, new inequalities for the case of non-ferromagnetic interactions are proved and a direct proof of a generalization of known magnetization properties is presented.

1. Introduction

Among the rigorous methods of examining models which exhibit phase transitions, an important role is played by the families of inequalities for correlation functions. GKS, GHS and FKG inequalities are used most frequently (Griffiths 1967, 1969, Kelly and Sherman 1968, Griffiths *et al* 1970, Fortuin *et al* 1971). It follows from GKS and FKG inequalities that mean values of some observables are directly related to interaction constants. This very important property is often used in various proofs. However, apart from in some specific cases, neither GKS nor FKG inequalities can tell us the net effect upon mean values when there is a simultaneous change of two groups of interaction constants in opposite directions. Also, they do not generally allow consideration of nonferromagnetic interactions. Holley (1974) proved a theorem which gives sufficient conditions for two probability measures (in an inequality form) when the mean value of some increasing observable is greater with respect to one of them than the other. The theorem naturally anticipates the net effect upon a mean value (for an increasing observable) of an increase of one group of interactions with a decrease of a second group. It also allows consideration of the case of non-ferromagnetic interactions. This paper presents results obtained by applying Holley's theorem to: (i) the lattice gas model; (ii) the Ising model; and (iii) higher order spin systems and continuum analogues of the spin-1 systems. Our main results are new inequalities for the non-ferromagnetic case and a direct proof of a generalization of known magnetization properties in the spin-$\frac{1}{2}$ Ising model.

2. Inequalities for the lattice gas model, Ising model and higher order spin systems

Let us recall for completeness the result of Holley.

Theorem (Holley)

Let Γ be a finite distributive lattice, x, $y \in \Gamma$, and let f be a real, increasing function on Γ

(i.e. if $x \geqslant y$ then $f(x) \geqslant f(y)$) which we will call observable. Also let ν_1 and ν_2 be two strictly positive probability measures on Γ. If

$$\nu_1(x \vee y)\nu_2(x \wedge y) \geqslant \nu_1(x)\nu_2(y) \tag{1}$$

for all $x, y \in \Gamma$, where $x \vee y (x \wedge y)$ denotes a least upper bound (greatest lower bound), then

$$\langle f \rangle_1 \equiv \sum_{x \in \Gamma} f(x)\nu_1(x) \geqslant \sum_{x \in \Gamma} f(x)\nu_2(x) \equiv \langle f \rangle_2. \tag{2}$$

Further we will consider only measures which correspond to the canonical ensembles of the systems under consideration. Hence

$$\nu_{\alpha,i}(x) = \frac{\exp(\lambda_{\alpha,i}(x))}{\Sigma_{x \in \Gamma} \exp(\lambda_{\alpha,i}(x))}$$

where $\lambda_{\alpha,i}(x) = -\beta \mathscr{H}_{\alpha,i}(x)$, $\mathscr{H}_{\alpha,i}$ is the Hamiltonian, $\alpha = 1, 2, 3$ denotes the model under consideration and $i = 1, 2$ denotes a possible choice of interaction constants in the Hamiltonian. Because of the specifications of $\nu_{\alpha,i}$, equation (1) can be rewritten as

$$\lambda_{\alpha,1}(x \vee y) + \lambda_{\alpha,2}(x \wedge y) \geqslant \lambda_{\alpha,1}(x) + \lambda_{\alpha,2}(y). \tag{1'}$$

2.1. Lattice gas model

Here, as with the finite distributive lattice Γ, we consider the set of subsets $A, B, P, \ldots$ of a finite set Λ,

$$\lambda_{1,i}(A) = \sum_{P \subset \Lambda} \varphi_i(P) n_P(A) \tag{3}$$

where

$$n(P) = \begin{cases} 1 & \text{if } A \supset P, P \neq \varnothing \\ 0 & \text{if } A \not\supset P \end{cases}$$

Lemma 1. The following conditions on $\varphi_i(P)$ are equivalent to (1): for any $a \in \Lambda$, $|a| = 1$ and any A, $B \subset \Lambda \setminus a$, such that $A \cap B = \varnothing$,

$$\sum_{P \subset A} \left(-\varphi_2(P + a) + \sum_{R \subset B} \varphi_1(P + R + a) \right) \geqslant 0 \tag{4}$$

where '+' denotes the symmetric difference of sets.

Proof. First one proves that (1') is equivalent to

$$\lambda_{\alpha,1}(A + B + a) - \lambda_{\alpha,1}(A + B) \geqslant \lambda_{\alpha,2}(A + a) - \lambda_{\alpha,2}(A) \tag{5}$$

for any $a \in \Lambda$, $|a| = 1$, A, $B \subset \Lambda \setminus a$, $A \cap B = \varnothing$ (for this proof, see appendix 1). Then one applies (3) to (5).

Consider now the case of one- and two-body interactions, i.e. $\varphi_i(P) = 0$ if $|P| > 2$. In this case (4) reduces to the following condition on interaction constants:

$$\varphi_1(a) - \varphi_2(a) + \sum_{b \in B} \varphi_1(a + b) + \sum_{c \in A} (\varphi_1(a + c) - \varphi_2(a + c)) \geqslant 0 \tag{6}$$

for any $a \in \Lambda$, $A, B \in \Lambda \setminus a$, $A \cap B = \varnothing$. Let f be an increasing observable throughout.

Corollary 1. Attractive interactions.

If

$$\varphi_2(a+c) \geqslant \varphi_1(a+c) \geqslant 0$$

and

$$\varphi_1(a) - \varphi_2(a) \geqslant \sum_{c \in A} (\varphi_2(a+c) - \varphi_1(a+c)) \tag{7}$$

then

$$\langle f \rangle_1 \geqslant \langle f \rangle_2.$$

Corollary 2. Non-attractive interactions.

For any $a, c \in \Lambda$, $a \neq c$, $\varphi(a+c) = \varphi_i(a+c)$ are of unspecified signs. Then $\langle f \rangle_1 \geqslant \langle f \rangle_2$ whenever

$$\varphi_1(a) - \varphi_2(a) + \min_{B \neq \varnothing} \left(0, \sum_{b \in B} \varphi(a+b)\right) \geqslant 0. \tag{8}$$

2.2. Ising model

In this case the lattice Γ is the same as in §2.1.

$$\lambda_{2,i}(A) = \sum_{P \subset \Lambda} J_i(P) \sigma_P(A) \tag{9}$$

where

$$\sigma_P = \prod_{a \in P} \sigma_a \qquad \text{and} \qquad \sigma_a = 2n_a - 1.$$

Lemma 2. The following conditions on $J_i(P)$ are equivalent to (1): for any $a \in \Lambda$, and any $B \subset A \subset \Lambda$, $B \subset \Lambda \diagdown a$

$$\sum_{R \subset B} 2^{|R|} \sum_{\substack{Q \\ Q \cap R + a = \varnothing}} J_1(Q+R+a) \sigma_Q(A) - \sum_{S \not\ni a} J_2(S+a) \sigma_S(A) \geqslant 0. \tag{10}$$

The proof is given in appendix 2.

Now let us restrict ourselves to the special case when at most pair interactions are present. Then (10) reduces to

$$J_1(a) - J_2(a) + \sum_{c \neq a} (J_1(c+a) - J_2(c+a)) \sigma_c(A) + 2 \sum_{b \in B} J_1(a+b) \geqslant 0 \tag{11}$$

for any $a \in \Lambda$, $B \subset A \subset \Lambda$, $a \notin B$.

Corollary 1. Ferromagnetic interactions.

Assume that for any $a, b \in \Lambda$ and $a \neq b$, $J_1(a+b) \geqslant 0$. Fulfillment of the inequality

$$J_1(a) - J_2(a) \geqslant \sum_{c \neq a} |J_1(c+a) - J_2(c+a)| \tag{12}$$

is sufficient for the inequality $\langle f \rangle_1 \geqslant \langle f \rangle_2$ to hold.

Corollary 2. Non-ferromagnetic interactions.

Let any $a, b \in \Lambda$, $a \neq b, J(a+b) = J_i(a+b)$ have arbitrary signs. Then $\langle f \rangle_1 \geq \langle f \rangle_2$ whenever

$$J_1(a) - J_2(a) + \min_{B \neq \varnothing}\left(0, 2 \sum_{b \in B} J(a+b)\right) \geq 0. \tag{13}$$

2.3. *Higher order spin systems*

First, let us define the underlying lattice Γ. Its elements form the set of all N-dimensional vectors x, y, whose components can take on the values $-p, -p+2, \ldots, p-2, p$, where p denotes a positive integer. We define spins S_i, $i = 1, 2, \ldots, N$ as the following functions on Γ: $S_i(x)$ is equal to the value of the ith component of x. With the help of the spins S_i we introduce the partial order that $x \geq y$ if and only if $S_i(x) \geq S_i(y)$ for all i. For the least upper bound $x \vee y$:

$$S_i(x \vee y) = \max[S_i(x), S_i(y)] \qquad \text{for all } i,$$

while for the greatest lower bound $x \wedge y$:

$$S_i(x \wedge y) = \min[S_i(x), S_i(y)] \qquad \text{for all } i.$$

The proof that defined Γ above in the distributive lattice is given in Lebowitz and Monroe (1972). The models under consideration are specified by

$$\lambda_{3,\kappa} = \sum_{i<j}^{n} \left(\tilde{\mathscr{J}}_\kappa(i,j) S_i(x) S_j(x) + \tilde{\gamma}_\kappa(i,j) S_i^2(x) S_j^2(x)\right) + \sum_{i=1}^{N} \left(\tilde{H}_\kappa(i) S_i(x) + \tilde{\mu}_\kappa(i) S_i^2(x)\right) \tag{14}$$

where $\tilde{\mathscr{J}}(i,j) = \tilde{\mathscr{J}}(j,i)$ and $\tilde{\gamma}_\kappa(i,j) = \tilde{\gamma}_\kappa(j,i)$.

Lemma 3. The following conditions upon interaction constants are sufficient for the fulfillment of the inequality (1):

$$2(\mathscr{J}(i,j) - 4(p-1)^2 |\gamma(i,j)|) + H_1(i) - H_2(i) - 2(p-1)|\mu_1(i) - \mu_2(i)| \geq 0 \tag{15}$$

and

$$H_1(i) - H_2(i) \geq 2(p-1)|\mu_1(i) - \mu_2(i)| \tag{16}$$

for any $i, j = 1, \ldots, N$, where $\mathscr{J}(i,j) = \frac{1}{2}\tilde{\mathscr{J}}(i,j)$, $\gamma(i,j) = \frac{1}{2}\tilde{\gamma}(i,j)$, $H(i) = \tilde{H}(i)/(N-1)$, $\mu(i) = \tilde{\mu}(i)/(N-1)$.

Proof. $\lambda_{3,\kappa}$ can be rewritten in the form of a double sum:

$$\lambda_{3,\kappa}(x) = \sum_{i \neq j}\sum \{\tfrac{1}{2}\tilde{\mathscr{J}}_\kappa(i,j) S_i(x) S_j(x) + \tfrac{1}{2}\tilde{\gamma}_\kappa(i,j) S_i^2(x) S_j^2(x) + [\tilde{H}_\kappa(i)/(N-1)] S_i(x) + [\tilde{\mu}_\kappa(i)/(N-1)] S_i^2(x)\}$$

and the inequality (1) will be established if it holds for each term in the sum. First assume that sites i and j are ordered in the states x and y, i.e. $S_i(x) \geq S_i(y)$ and $S_j(x) \geq S_j(y)$ or $S_i(x) \leq S_i(y)$ and $S_j(x) \leq S_j(y)$. Then we get $\mathscr{J}_1(i,j) = \mathscr{J}_2(i,j)$ and $\gamma_1(i,j) = \gamma_2(i,j)$ and $H_1(i) - H_2(i) \geq 2(p-1)|\mu_1(i) - \mu_2(i)|$. In the second step, assume that the sites i and j are unordered in the states x and y. Because of the symmetry between i and j it is enough to consider only one choice, for example $S_i(x) < S_i(y)$ and $S_j(x) > S_j(y)$. This leads to the condition

$$2[\mathscr{J}(i,j) - 4(p-1)^2|\gamma(i,j)|] + H_1(i) - H_2(i) - 2(p-1)|\mu_1(i) - \mu_2(i)| \geq 0.$$

Remark. Note that in the case $p = 1$ (15) and (16) reduce to the corresponding Ising model condition and when $\mu_1(i) = \mu_2(i)$ we get the result proved by Lebowitz and Monroe (1972).

Finally let us consider continuum models of particles of two species with the following fugacity

$$z_\kappa^\alpha(\boldsymbol{r}) = \exp \beta(\mu_\kappa(\boldsymbol{r})\alpha^2 + H_\kappa(\boldsymbol{r})\alpha) \tag{17}$$

where $\alpha = -2, 2$, $\boldsymbol{r} \in V \subset R^\nu$, $\kappa = 1, 2$ and the following pair interactions

$$\Phi_\kappa^{\alpha\alpha'}(|\boldsymbol{r} - \boldsymbol{r}'|) = -\mathscr{J}_\kappa(|\boldsymbol{r} - \boldsymbol{r}'|)\alpha\alpha' - \gamma_\kappa(|\boldsymbol{r} - \boldsymbol{r}'|)\alpha^2\alpha'^2. \tag{18}$$

Lebowitz and Monroe (1972) have shown that models defined by (17) and (18) can be thought of as zero lattice spacing limits, $\xi \to 0$, of the models defined by (14) for $p = 2$. The corresponding Hamiltonians are related by the following equations:

$$\mathscr{J}(i,j) = \mathscr{J}(|\boldsymbol{r}_i - \boldsymbol{r}_j|), \qquad \gamma(i,j) = \gamma(|\boldsymbol{r}_i - \boldsymbol{r}_j|),$$
$$H(i) = H(\boldsymbol{r}_i), \qquad \mu(i) = \mu(\boldsymbol{r}_i) + \beta^{-1} \ln \xi.$$

Let $S_c(X; \alpha, \omega)$ denotes the function equal to the number of particles of species α which are in an open set $\omega \subset V$ in configuration X. Also let $s_\xi(x; \alpha, \omega)$ be defined by the equation $s_\xi(x; \alpha, \omega) = \Sigma_{i \in \omega} \delta(S_i(x), \alpha)$, where $\delta(\cdot)$ denotes the Kronecker function. The above remark and lemma 3 lead to the following corollary.

Corollary 1.

Let f be a real continuous function and let the observable $f[s_\xi(x; \alpha, \omega)]$ be increasing. The conditions

$$2[\mathscr{J}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) - 4|\gamma(|\boldsymbol{r}_i - \boldsymbol{r}_j|)|] + H_1(\boldsymbol{r}_i) - H_2(\boldsymbol{r}_i) - 2|\mu_1(\boldsymbol{r}_i) - \mu_2(\boldsymbol{r}_i)| \geq 0 \tag{19}$$

and

$$H_1(\boldsymbol{r}_i) - H_2(\boldsymbol{r}_i) \geq 2|\mu_1(\boldsymbol{r}_i) - \mu_2(\boldsymbol{r}_i)| \tag{20}$$

are sufficient for the validity of the following inequality:

$$\langle f[s_c(X; \alpha, \omega)]\rangle_{c,1} \geq \langle f[s_c(X; \alpha, \omega)]\rangle_{c,2} \tag{21}$$

where $\langle \cdots \rangle_{c,\kappa}$ stands for the mean value with respect to the continuum system grand ensemble.

3. Final remarks

Some explanations in connection with our statements in § 1 are necessary. First, in fact, Lebowitz (1972) succeeded in deriving the net effect upon magnetization of a decrease of pair interactions in conjunction with a suitable increase of magnetic field. This has been done by combining GKS and FKG inequalities. Holley's theorem seems to be a natural starting point for derivation of such inequalities as it allows direct proofs to be given. For example, consider a spin-$\frac{1}{2}$ Ising model with pair interactions

$$\lambda_2(A)=\sum_{\substack{P\subset\Lambda\\|P|\leq 2}} J(P)\sigma_P(A)=\sum_{\substack{a,b\\a\neq b}} J(a+b)\sigma_{a,b}(A)+\sum_{a\in\Lambda} h(a)\sigma_a(A).$$

Let us now introduce some new notation. $\{J\}$ denotes the set of all $J(a+b)$ and $\{h\}$ the set of all $h(a)$, $a, b\in\Lambda$, $a\neq b$, for any increasing observable

$$\langle f\rangle\equiv\frac{\Sigma_{P\subset\Lambda} f(P)\, e^{\lambda_2(P)}}{\Sigma_{P\subset\Lambda} e^{\lambda_2(P)}}\equiv\mathscr{F}_\Lambda(\{J\},\{h\}).$$

Suppose that every pair interaction constant $J(a+b)$ has been diminished by the positive value $\delta J(a+b)$ such that $J(a+b)-\delta J(a+b)\geq 0$ and at the same time every $h(a)$ has been enlarged by $\delta h(a)=\Sigma_{b\in\Lambda,b\neq a}\,\delta J(a+b)$. The following inequality is an immediate consequence of corollary 1 of § 2.2:

$$\mathscr{F}_\Lambda(\{J-\delta J\},\{h+\delta h\})\geq\mathscr{F}_\Lambda(\{J\},\{h\}) \tag{22}$$

or changing $J\to J+\delta J$

$$\mathscr{F}_\Lambda(\{J\},\{h+\delta h\})\geq\mathscr{F}_\Lambda(\{J+\delta J\},\{h\}) \tag{23}$$

(where $\delta h=\Sigma_{b\in\Lambda,b\neq a}\,\delta J(a+b)$). For translationally invariant, stable interactions, after passing to the limit $\Lambda\to\infty$, we get

$$\mathscr{F}(\{J-\delta J\}, h+\delta h)\geq\mathscr{F}(\{J\}, h) \tag{22$'$}$$

or

$$\mathscr{F}(\{J\}, h+\delta h)\geq\mathscr{F}(\{J+\delta J\}, h) \tag{23$'$}$$

where $\mathscr{F}(\{J\}, h)=\lim_{\Lambda\to\infty}\mathscr{F}_\Lambda(\{J\}, h)$. Second, we would like to mention that Lebowitz (1971) derived inequalities for a specific antiferromagnetic model, starting from FKG inequalities and using a suitable transformation of spins. Some applications of the derived inequalities are in preparation.

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Appendix 1. The proof of the equivalence between (1′) and (5)

First we show that (5) follows from (1′). (1′) reads: for any two subsets R, S, of Λ

$$\lambda_{\alpha,1}(R\cup S)+\lambda_{\alpha,2}(R\cap S)\geq\lambda_{\alpha,1}(R)+\lambda_{\alpha,2}(S).$$

But any R, $S\subset\Lambda$ can be represented by three disjoint sets $A=R\cap S$, $B=R\setminus A$, $C=S\setminus A$. (1′) is equivalent to

$$\lambda_{\alpha,1}(A+B+C)-\lambda_{\alpha,1}(A+B)\geq\lambda_{\alpha,2}(A+C)-\lambda_{\alpha,2}(A) \tag{A.1}$$

for any A, B, $C \subset \Lambda$ which are pairwise disjoint. Taking $C = a$ we get (5):

$$\lambda_{\alpha,1}(A+B+a) - \lambda_{\alpha,1}(A+B) \geq \lambda_{\alpha,2}(A+a) - \lambda_{\alpha,2}(A).$$

We now prove that (1′) follows from (5). Any $C \subset \Lambda$ disjoint with A and B can be written as $a_1 + a_2 + \ldots a_N$, $a_1, \ldots, a_N \in \Lambda \setminus (B \cup A)$. To prove the converse we apply (5) several times and obtain the following sequence of inequalities:

$$\lambda_{\alpha,1}(A+B+a_1) - \lambda_{\alpha,1}(A+B) \geq \lambda_{\alpha,2}(A+a_1) - \lambda_{\alpha,2}(A)$$

$$\lambda_{\alpha,1}(A+a_1+B+a_2) - \lambda_{\alpha,1}(A+a_1+B) \geq \lambda_{\alpha,2}(A+a_1+a_2) - \lambda_{\alpha,2}(A+a_1)$$

$$\lambda_{\alpha,1}(A+a_1+a_2+B+a_3) - \lambda_{\alpha,1}(A+a_1+a_2+B)$$
$$\geq \lambda_{\alpha,2}(A+a_1+a_2+a_3) - \lambda_{\alpha,2}(A+a_1+a_2)$$

$$\vdots$$

$$\lambda_{\alpha,1}(A+C\setminus a_N+B+a_N) - \lambda_{\alpha,1}(A+C\setminus a_N+B)$$
$$\geq \lambda_{\alpha,2}(A+C\setminus a_N+a_N) - \lambda_{\alpha,2}(A+C\setminus a_N).$$

After addition of these inequalities we obtain (A.1).

Appendix 2. The proof of lemma 2

Because (4) is equivalent to (1), we prove that (10) is equivalent to (4). For this purpose we use the expression relating lattice gas interaction constants $\varphi_i(P)$ with Ising model interaction constants $J_i(P)$:

$$\varphi_i(P) = \sum_{R \supset P} 2^{|P|}(-1)^{|R|-|P|} J_i(R) \tag{A.2}$$

which follows from the requirement $\lambda_{1,i} = \lambda_{2,i}$. So for any $A \subset \Lambda \setminus (B+a)$ and any $R \subset B \subset \Lambda \setminus a$, $a \in \Lambda$:

$$\sum_{P \subset A} \varphi_1(P+R+a)$$

$$= \sum_{P \subset A} \sum_{S \supset P+R+a} 2^{|R+P+a|}(-1)^{|S|-|R+P+a|} J_1(S)$$

$$= \sum_{P \subset A} \sum_{\substack{Q \supset P \\ Q \cap R+a=\varnothing}} 2^{|R+P+a|}(-1)^{|Q|-|P|} J_1(Q+R+a)$$

$$= \sum_{\substack{Q' \\ Q' \cap R+a=\varnothing}} \sum_{P \subset RA \cap Q'} 2^{|R+P+a|}(-1)^{|Q'|-|P|} J_1(Q'+R+a)$$

$$= 2\,.\,2^{|R|} \sum_{\substack{Q' \\ Q' \cap R+a=\varnothing}} J_1(Q'+R+a)(-1)^{|Q'|+|A \cap Q'|}$$

$$= 2\,.\,2^{|R|} \sum_{\substack{Q' \\ Q' \cap R+a=\varnothing}} J_1(Q'+R+a)\sigma_{Q'}(\Lambda \setminus A) \tag{A.3}$$

and similarly we get

$$\sum_{P\subset A} \varphi_2(P+a) = 2 \sum_{Q \not\ni a} J_1(Q+a)\sigma_Q(\Lambda\setminus A). \tag{A.4}$$

(A.3) and (A.4) hold for any $A \subset \Lambda\setminus B$. Changing the notation from A to $\Lambda\setminus A$ (4) can be rewritten in the following form:

$$\sum_{R\subset B} 2^{|R|} \sum_{\substack{Q' \\ Q'\cap R+a=\varnothing}} J_1(Q'+R+a)\sigma_{Q'}(A) - \sum_{Q\not\ni a} J_2(Q+a)\sigma_Q(A) \geqslant 0$$

and this inequality should hold for any $A \supset B$.

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